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Closed Operator Ideals and Interpolation

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We consider closed operator ideals, which mean operator ideals A whose components $A(E, F)$ are closed subspaces of the space $L(E, F)$. Using interpolation techniques, we obtain general results on products of closed ideals. Furthermore, we investigate which closed ideals A possess the factorization property, i.e., each operator of A factors through a space with the related property "A." Applications of these results yield the answer to some open questions in ideal theory.

INTRODUCTION

In this paper we consider operator ideals A whose components $A(E, F)$ are closed subspaces of $L(E, F)$. These ideals are called closed. Most of the classical operator ideals, e.g., the ideals of compact, weakly compact, completely continuous operators, are closed. Products and quotients of classical closed operator ideals were considered by Pietsch [12, 13] and Puhl [15]. The aim of this paper is to prove several general results on closed operator ideals. Our main tool is the interpolation technique [10], which has proved to be useful in various aspects of operator ideal theory (cf. [4, 11, 12]).

In Section I we deal with products of operator ideals. We prove that the product of closed ideals is closed, as well, thus answering a question of Pietsch [14]. Furthermore, we give a convenient description of products of certain classes of ideals. This result yields, in particular, representations of two concrete products which were left open by Pietsch in [13].

Section II is devoted to closed ideals with the factorization property, which states that every operator of the ideal A can be factored through a Banach space with the corresponding property "A." Factorization theorems were first obtained by Davis *et al.* [6], and later by Beauzamy [2–4]. We give here a sufficient condition for general closed operator ideals to possess the factorization property. This theorem unifies the corresponding results of [2–4, 6] and yields a new factorization concerning dual Radon–Nikodým operators. We also answer

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a question of Beauzamy [4] about the Banach–Saks property of interpolation spaces. Some counterexamples will show that the assumptions of the general factorization theorem are essential.

Definitions and Notation

Let E and F be Banach spaces. E' denotes the dual of E, B_E the unit ball of E, and I_E the identity map of E. $L(E,F)$ is the space of bounded linear operators equipped with the usual operator norm. For $T \in L(E,F)$, T' denotes the dual of T. Given $1 \leqslant p < \infty$ and a sequence of Banach spaces (E_n) $(n = 1, 2,...)$, we denote the space of sequences (x_n) with $x_n \in E_n$ and $\|(x_n)\| = (\sum_n \| x_n \|^p)^{1/p} < \infty$ by $(\sum E_n)_p$.

An operator ideal A is a class of bounded linear operators such that the components $A \cap L(E,F) = A(E,F)$ satisfy the following three conditions: (i) $A(E,F)$ is a linear subset of $L(E,F)$. (ii) $A(E,F)$ contains the finite rank operators. (iii) $R \in L(E_1, E)$, $S \in A(E,F)$, and $T \in L(F,F_1)$ imply $TSR \in A(E_1, F_1)$.

Remark that, if A contains a nonzero operator, then (ii) is already a consequence of (i) and (iii). The notion of an operator ideal is a natural generalization of the ring-theoretical ideal concept. It allows one to deal with operators between different Banach spaces and includes a comparison of different components $A(E,F)$ and $A(E_1, F_1)$. For the theory of operator ideals we refer to [8, 11, 12].

An operator ideal A is called closed if the components $A(E,F)$ are closed subspaces of $L(E,F)$. Diverse examples of closed operator ideals are given in [12, 13]. Some of them will be mentioned below. New closed operator ideals also appeared in connection with the concept of superideals (cf. [1, 9]). Given two operator ideals A and B, the product $B \circ A$ is defined in the following way: $T \in L(E,F)$ belongs to $B \circ A$ if there is a Banach space G and operators $T_1 \in A(E, G)$, $T_2 \in B(G,F)$ such that T admits the representation $T = T_2 T_1$. It should be emphasized that the Banach space G is arbitrary; thus this product concept differs essentially from the ring-theoretical one. An operator ideal A is called injective if for every isomorphic embedding $J \in L(F,F_1)$ the following holds: $T \in L(E,F)$ and $JT \in A(E,F_1)$ imply $T \in A(E,F)$. The operator ideal A is surjective if for every surjection $Q \in L(E_1, E)$, $T \in L(E,F)$, and $TQ \in A(E_1, F)$ imply $T \in A(E,F)$. Finally, every operator ideal A defines a class of Banach spaces Space(A) in the following way: $E \in \text{Space}(A)$ iff $I_E \in A(E, E)$.

We also need some notions from interpolation theory (cf. [5, 10]). Let (E_1, E_2) be an interpolation pair, i.e., E_1 and E_2 are continuously embedded into a topological vector space. Define, as usual, the norms on $E_1 \cap E_2$ and $E_1 + E_2$ by setting

$$\| x \|_{E_1 \cap E_2} = \max(\| x \|_{E_1}, \| x \|_{E_2}) \qquad (x \in E_1 \cap E_2)$$

and

$$\| x \|_{E_1+E_2} = \inf_{x=x_1+x_2} (\| x_1 \|_{E_1} + \| x_2 \|_{E_2}) \qquad (x \in E_1 + E_2).$$

Now let E be an intermediate space between E_1 and E_2, i.e., $E_1 \cap E_2 \subset E \subset E_1 + E_2$. Then E is said to be of class $J(\theta, E_1, E_2)$, where $0 < \theta < 1$, if there exists a constant C such that for all $t > 0$ and $x \in E_1 \cap E_2$,

$$\| x \|_E \leqslant Ct^{-\theta} \max(\| x \|_{E_1}, t \| x \|_{E_2}).$$

E is of class $K(\theta, E_1, E_2)$ if there is a constant C such that for all $t > 0$ and $x \in E$,

$$\| x \|_E \geqslant Ct^{-\theta} \inf\{\| x_1 \|_{E_1} + t \| x_2 \|_{E_2} : x_1 \in E_1, x_2 \in E_2, x = x_1 + x_2\}.$$

Finally, we recall two of the (up to an isomorphism) equivalent definitions of the Lions–Peetre interpolation spaces $(E_1, E_2)_{\theta,p}$, where $0 < \theta < 1$ and $1 < p < \infty$: Let ξ_1 and ξ_2 be such that $\xi_1(\xi_1 - \xi_2)^{-1} = \theta$. Then $(E_1, E_2)_{\theta,p}$ is the space of all $x \in E_1 + E_2$ which can be represented as

$$x = \sum_{n=-\infty}^{+\infty} x_n \qquad (x_n \in E_1 \cap E_2)$$

with

$$\| x \| = \inf_{x=\sum x_n} \max \left(\left(\sum \| e^{\xi_1 n} x_n \|_{E_1}^p \right)^{1/p}, \left(\sum \| e^{\xi_2 n} x_n \|_{E_2}^p \right)^{1/p} \right) < \infty.$$

The second definition is: $(E_1, E_2)_{\theta,p}$ is the space of all $x \in E_1 + E_2$ which admit representations of the form $x = x_{1n} + x_{2n}$, $x_{1n} \in E_1$, $x_{2n} \in E_2$ ($n = 0, \pm 1, \ldots$) with

$$\| x \| = \inf_{x=x_{1n}+x_{2n}} \max \left(\left(\sum \| e^{\xi_1 n} x_{1n} \|_{E_1}^p \right)^{1/p}, \left(\sum \| e^{\xi_2 n} x_{2n} \|_{E_2}^p \right)^{1/p} \right) < \infty.$$

The equivalence of both definitions was proved in [10, p. 18]. Note that $(E_1, E_2)_{\theta,p}$ is of class $J(\theta, E_1, E_2)$ and $K(\theta, E_1, E_2)$.

I. Products of Closed Operator Ideals

Our first theorem shows that the class of closed operator ideals is stable under products. This answers a question of Pietsch [14].

Theorem 1.1. *If A and B are closed operator ideals, then $B \circ A$ is closed, as well.*

We need the following

LEMMA 1.2. *Let $T \in L(E, F)$, $T \neq 0$, and suppose $T = T_2 T_1$, where $T_1 \in L(E, G)$ and $T_2 \in L(G, F)$. Then there is an equivalent norm $\| \ \|_0$ on G such that*

$$\| T_1 \|_{L(E, G_0)} \| T_2 \|_{L(G_0, F)} = \| T \|,$$

where $G_0 = (G, \| \ \|_0)$.

Proof. Define

$$\| z \|_0 = \max(\| T \| \| T_1 \|^{-1} \| z \|_G , \| T_2 z \|_F) \qquad (z \in G).$$

Then we have

$$\| T \| \| T_1 \|^{-1} \| z \| \leqslant \| z \|_0 \leqslant \| T_2 \| \| z \|.$$

It follows furthermore that

$$\| T_2 \|_{L(G_0, F)} \leqslant 1.$$

Finally, we get for $x \in B_E$,

$$\| T_1 x \|_0 = \max(\| T \| \| T_1 \|^{-1} \| T_1 x \|_G , \| T_2 T_1 x \|_F) \leqslant \| T \|.$$

Therefore

$$\| T_1 \|_{L(E, G_0)} \leqslant \| T \|,$$

which yields the desired result.

Proof of Theorem 1.1. It is sufficient to prove that $T_n \in B \circ A(E, F)$ $(n = 1, 2, \ldots)$ and $\sum \| T_n \| < \infty$ imply $T = \sum T_n \in B \circ A(E, F)$. According to Lemma 1.2, each operator T_n can be represented as a product $T_n = T_{2n} T_{1n}$ with $T_{1n} \in A(E, G_n)$, $T_{2n} \in B(G_n , F)$, and $\| T_{1n} \| \| T_{2n} \| = \| T_n \|$. Multiplying, if necessary, T_{1n} and T_{2n} by suitable scalars, we may assume

$$\| T_{1n} \| = \| T_{2n} \| = \| T_n \|^{1/2}.$$

Now let $G = (\sum G_n)_2$ and define $T_1 \in L(E, G)$ and $T_2 \in L(G, F)$ by setting

$$T_1 x = (T_{1n} x)_{n=1}^{\infty} , \qquad T_2 (z_n)_{n=1}^{\infty} = \sum_{n=1}^{\infty} T_{2n} z_n .$$

Then $T = T_2 T_1$, and we get, since A and B are closed, $T_1 \in A(E, G)$ and $T_2 \in B(G, F)$, thus $T \in B \circ A(E, F)$.

In the sequel we shall consider the product of closed operator ideals which have special properties. Our aim is to prove the following representation theorem.

THEOREM 1.3. *Let A and B be closed operator ideals. If A is injective and B is surjective, then*

$$B \circ A = B \cap A,$$

where $B \cap A$ denotes the intersection of both ideals.

First we mention without proof the following two lemmas, which are easy consequences of the properties of injective and surjective ideals (cf. [11, pp. 31–38] or [12, 4.6–4.7]).

LEMMA 1.4. *Let E, F_1, and F_2 be Banach spaces, let A be an injective operator ideal, and let $T_1 \in A(E, F_1)$, $T_2 \in L(E, F_2)$. If there exists a constant C such that*

$$\| T_2 x \|_{F_2} \leqslant C \| T_1 x \|_{F_1} \qquad (x \in E)$$

then $T_2 \in A(E, F_2)$.

LEMMA 1.5. *Let E_1, E_2, and F be Banach spaces, let B be a surjective operator ideal, and let $T_1 \in B(E_1, F)$, $T_2 \in L(E_2, F)$. If there is a constant C such that*

$$T_2(B_{E_2}) \subset C T_1(B_{E_1}),$$

then $T_2 \in B(E_2, F)$.

The following two propositions of interpolation type are the crucial point of the proof of Theorem 1.3.

PROPOSITION 1.6. *Let A be an injective closed operator ideal and let E, F_1, F_2, and F be Banach spaces. Suppose (F_1, F_2) is an interpolation pair and F is of class $J(\theta, F_1, F_2)$ $(0 < \theta < 1)$. If $T \in L(E, F_1)$ and $T \in A(E, F_2)$, then $T \in A(E, F)$.*

Proof. It follows immediately from the interpolation hypothesis that for each natural number n there is a constant C_n such that

$$\| Tx \|_F \leqslant \max(n^{-1} \| Tx \|_{F_1}, C_n \| Tx \|_{F_2}) \qquad (x \in E).$$

Put $W = T'(B_{F'})$. Using the bipolar theorem and the weak-star compactness of the sets $T'(B_{F_1'})$ and $T'(B_{F_2'})$, we conclude

$$W \subset \operatorname{co}(n^{-1} T'(B_{F_1'}) \cup C_n T'(B_{F_2'})) \subset n^{-1} T'(B_{F_1'}) + C_n T'(B_{F_2'}),$$

where co denotes the convex hull. By the above inclusion we can find maps φ_n and ψ_n from W into $n^{-1}T'(B_{F_1'})$ and $C_n T'(B_{F_2'})$, respectively, with

$$\varphi_n(w) + \psi_n(w) = w \qquad (w \in W).$$

We now consider the space $l_\infty(W)$ of bounded functions on W and define operators R_n, $S_n \in L(E, l_\infty(W))$ by setting

$$R_n x = (\langle x, \varphi_n(w)\rangle)_{w \in W}, \qquad S_n x = (\langle x, \psi_n(w)\rangle)_{w \in W}.$$

Let F_0 be the closure of $\operatorname{Im} T$ in F, and let J be the canonical isometric embedding of F_0 into $l_\infty(W)$ defined by

$$J(Tx) = (\langle x, w\rangle)_{w \in W} \qquad (Tx \in \operatorname{Im} T).$$

It follows that

$$R_n + S_n = JT_0,$$

where T_0 is the operator T, considered as an operator from E into F_0. Furthermore,

$$\| S_n x \| \leqslant C_n \sup_{g \in B_{F_2'}} |\langle x, T'g\rangle| = C_n \| Tx \|_{F_2}.$$

We now deduce from Lemma 1.4 that $S_n \in A(E, l_\infty(W))$. Finally, we have

$$\| R_n \| \leqslant n^{-1} \| T \|_{L(E, F_1)};$$

thus,

$$\lim_{n \to \infty} \| JT_0 - S_n \| = 0,$$

and, since A is closed, it follows that

$$JT_0 \in A(E, l_\infty(W)).$$

Using the injectivity of A we get $T_0 \in A(E, F_0)$ and $T \in A(E, F)$.

The next proposition is, in a certain sense, dual to the preceding one.

PROPOSITION 1.7. *Let B be a surjective closed operator ideal and let E, E_1, E_2, and F be Banach spaces. Suppose (E_1, E_2) is an interpolation pair and E is of class $K(\theta, E_1, E_2)$ $(0 < \theta < 1)$. If $T \in L(E_1, F)$ and $T \in B(E_2, F)$ then $T \in B(E, F)$.*

Proof. It follows from the hypothesis that for each n there exists a C_n such that

$$\| x \|_E \geqslant \inf_{x = x_1 + x_2} (n \| x_1 \|_{E_1} + C_n^{-1} \| x_2 \|_{E_2}) \qquad (x \in E).$$

Setting $T(B_E) = W$, we get

$$W \subset 2\,\mathrm{co}(n^{-1}T(B_{E_1}) \cup C_n T(B_{E_2})) \subset 2n^{-1}T(B_{E_1}) + 2C_n(T(B_{E_2})).$$

As in the proof of Proposition 1.6 we find mappings φ_n and ψ_n from W into $2n^{-1}T(B_{E_1})$ and $2C_nT(B_{E_2})$, respectively, with

$$\varphi_n(w) + \psi_n(w) = w \qquad (w \in W).$$

Consider the space $l_1(W)$ of absolutely summable functions on the set W and let T_0 be the linear operator from $l_1(W)$ into F given by $T_0 e_w = w$ $(w \in W)$, where e_w is the corresponding unit vector. Remark that $T_0(B_{l_1(W)}) = W$. Now we define operators R_n, $S_n \in L(l_1(W), F)$ by setting

$$R_n e_w = \varphi_n(w), \qquad S_n e_w = \psi_n(w) \qquad (w \in W).$$

Then we get

$$R_n + S_n = T_0, \qquad \| R_n \| \leqslant 2n^{-1} \| T \|_{L(E_1, F)} \cdot$$

and

$$S_n(B_{l_1(W)}) \subset 2C_nT(B_{E_2}).$$

It follows by Lemma 1.5 that $S_n \in B(l_1(W), F)$. Furthermore, we have

$$\lim_{n\to\infty} \| S_n - T_0 \| = 0.$$

Therefore $T_0 \in B(l_1(W), F)$, and, since

$$T_0(B_{l_1(W)}) = W = T(B_E),$$

we conclude again by Lemma 1.5 that $T \in B(E, F)$.

Proof of Theorem 1.3. Suppose $T \in B(E, F) \cap A(E, F)$. Put $G_1 = E/\mathrm{Ker}\, T$ and $G_2 = F$. Let Q be the quotient map of E onto G_1 and let J be the injection of G_1 into G_2 induced by T. Now take any space G which is of both class $J(\theta, G_1, G_2)$ and class $K(\theta, G_1, G_2)$ with $0 < \theta < 1$, e.g., $G = (G_1, G_2)_{\theta,p}$. Denote the corresponding embedding maps of G_1 into G and of G into G_2 by J_1 J_2, respectively. Then we have $T = JQ \in A(E, G_2)$, and we get from Proposition 1.6 that $J_1Q \in A(E, G)$. Furthermore, the surjectivity of B yields $J \in B(G_1, G_2)$. This together with Proposition 1.7 implies $J_2 \in B(G, G_2)$. Thus, we have shown that

$$T = J_2J_1Q \in B \circ A(E, F),$$

and therefore

$$B \cap A \subset B \circ A.$$

The converse inclusion is trivial.

Theorem 1.3 yields simple descriptions of some products of closed ideals. First recall that an operator is said to be completely continuous if it transforms weakly convergent sequences into norm convergent sequences. An operator is called separable if it has a separable range. The following corollary fills up two gaps of the multiplication table given by Pietsch in [13].

COROLLARY 1.8. *Let W, V, X be the ideals of weakly compact, completely continuous, and separable operators, respectively. Then*

$$W \circ V = W \cap V$$

and

$$W \circ X = W \cap X.$$

Another corollary concerns the idempotence of certain ideals. An operator ideal is called idempotent if $A \circ A = A$.

COROLLARY 1.9. *Each injective, surjective, closed operator ideal is idempotent.*

Clearly, every operator ideal with the factorization property (see below) is idempotent. But it should be mentioned that the class of idempotent ideals is essentially larger. For example, the ideals or compact operators, uniformly convexifying operators, and operators of type Rademacher are closed, injective, and surjective, but do not possess the factorization property (cf. [1, 2]).

II. FACTORIZATION THROUGH BANACH SPACES

An operator ideal A is said to have the factorization property if for every operator $T \in A(E, F)$ there exists a Banach space $G \in \mathrm{Space}(A)$ and operators $T_1 \in L(E, G)$ and $T_2 \in L(G, F)$ such that $T = T_2 T_1$. In other terms, every operator of the ideal A factors through a space with the corresponding Banach space property "A." First we give general sufficient conditions for an ideal to possess the factorization property. For this we introduce the following notion.

Let $1 \leqslant p < \infty$. An operator ideal A satisfies the $\sum_p$-condition iff for arbitrary Banach spaces E_n, F_n $(n = 1, 2, \ldots)$ the following holds.

$$\left(\sum_p\right): \text{If } T \in L\left(\left(\sum E_n\right)_p, \left(\sum F_n\right)_p\right), \text{ and } Q_n T P_m \in A(E_m, F_n)$$

$$(m, n = 1, 2, \ldots), \text{ then } T \in A\left(\left(\sum E_n\right)_p, \left(\sum F_n\right)_p\right).$$

Here P_m and Q_n denote the projections of $(\sum E_n)_p$, $(\sum F_n)_p$ onto the coordinates E_m and F_n, respectively. In other words, the condition states that the operator T belongs to A if all elements of its matrix representation belong to A. An argument similar to the proof of Theorem 1.1 shows that every ideal which satisfies the $\sum_p$-condition is closed. We now state the general factorization theorem.

THEOREM 2.1. *Let $1 < p < \infty$ and let A be an injective and surjective operator ideal which satisfies the $\sum_p$-condition. Then A possesses the factorization property.*

We shall derive this theorem from the following.

PROPOSITION 2.2. *Let $1 < p < \infty$, $0 < \theta < 1$, and let A be an injective and surjective operator ideal which satisfies the $\sum_p$-condition. Suppose (E_1, E_2) is an interpolation pair, and let I be the canonical injection of $E_1 \cap E_2$ into $E_1 + E_2$. If*

$$I \in A(E_1 \cap E_2, E_1 + E_2),$$

then

$$(E_1, E_2)_{\theta,p} \in \operatorname{Space}(A).$$

Proof. Denote $E = E_1 \cap E_2$, $F = E_1 + E_2$, $G = (E_1, E_2)_{\theta,p}$, and define the following equivalent norms on E and F, respectively:

$$\| x \|_n = \max(\| e^{\xi_1 n} x \|_{E_1}, \quad \| e^{\xi_2 n} x \|_{E_2}) \qquad (x \in E,\ n = 0, \pm 1, \ldots),$$

$$\| y \|_n = \inf_{y = y_1 + y_2} (\| e^{\xi_1 n} y_1 \|_{E_1} + \| e^{\xi_2 n} y_2 \|_{E_2}) \qquad (y \in F,\ n = 0, \pm 1, \ldots).$$

Denote $E_n = (E, \| \, \|_n)$ and $F_n = (F, \| \, \|_n)$. Using, respectively, the two discrete characterizations of $(E_1, E_2)_{\theta,p}$ mentioned in the preliminaries (cf. also [10, p. 18]) and the equivalence of norms on the two-dimensional space, it follows immediately that G is isomorphic to the two spaces

$$G_1 = \left\{ y \in F \colon y = \sum_{n=-\infty}^{+\infty} x_n,\ x_n \in E_n,\ \| y \|_1 = \inf_{y = \sum x_n} \left(\sum_{n=-\infty}^{+\infty} \| x_n \|_n^p \right)^{1/p} < \infty \right\},$$

$$G_2 = \left\{ y \in F \colon \| y \|_2 = \left(\sum_{n=-\infty}^{+\infty} \| y \|_n^p \right)^{1/p} < \infty \right\}.$$

In other words, there is a canonical surjection Q of $(\sum E_n)_p$ onto G defined by

$$Q(x_n) = \sum_{n=-\infty}^{+\infty} x_n,$$

and a canonical isomorphic embedding J of G into $(\sum F_n)_p$ defined by

$$Jz = (z) = (..., z, z, z, ...).$$

Obviously, $Q_n JQP_m$ is the canonical injection of E_m into F_n; therefore it belongs to A. Using the $\sum_p$-condition, we obtain

$$JI_G Q = JQ \in A.$$

On the other hand, injectivity and surjectivity yield $I_G \in A$; thus $G \in \text{Space}(A)$.

Proof of Theorem 2.1. Let $T \in A(E, F)$ and denote $G_1 = E/\text{Ker } T$, $G_2 = F$. Then, by surjectivity, the injection of G_1 into G_2 induced by T belongs to A. We conclude from Proposition 2.2 that $G = (G_1, G_2)_{\theta,p} \in \text{Space}(A)$ and obtain the desired factorization of T: $E \to E/\text{Ker } T \to G \to F$.

The following theorem yields a list of ideals possessing the factorization property. Let us first recall the necessary definitions. An operator $T \in L(E, F)$ is called a Rosenthal operator if for each $S \in L(l_1, E)$, the composition TS is not an isomorphic embedding. Using Rosenthal's theorem [17], we get the following characterization: T is a Rosenthal operator iff it maps bounded sequences into sequences possessing weak Cauchy subsequences. T is said to be a Banach-Saks operator if it maps bounded sequences into sequences possessing Cesaro convergent subsequences. $T \in L(E, F)$ is called a Radon–Nikodým operator if it maps each μ-continuous E-valued measure of finite variation into a μ-differentiable F-valued measure, where μ is an arbitrary probability measure. Finally, $T \in L(E, F)$ is called decomposing if for every probability space (Ω, μ) and every operator $S \in L(F, L_\infty(\Omega, \mu))$ there is a μ-measurable E'-valued function $a(\omega)$ such that

$$STx = \langle x, a(\cdot)\rangle \qquad (x \in E).$$

The significance of this ideal becomes clear by the following result (cf. [12, 24.4.3]): T is decomposing if and only if its dual T' is a Radon–Nikodým operator.

THEOREM 2.3. *The following ideals are injective and surjective and satisfy the $\sum_p$-condition for $1 < p < \infty$. Consequently, they possess the factorization property.*

(i) *Weakly compact operators.*
(ii) *Rosenthal operators.*
(iii) *Banach–Saks operators.*
(iv) *Decomposing operators.*

Parts of this result are known and stated for completeness: In case (i) the factorization property was proved by Davis *et al.* [6]; in cases (ii) and (iii), by Beauzamy [2, 3].

Proof of Theorem 2.3. We restrict ourselves to proofs of (iii) and (iv). It is shown in [10, 24.5.5] that the ideal of decomposing operators is injective and surjective, while the ideal of Banach–Saks operators obviously possesses these properties. The following observation is useful for the verification of the $\sum_p$-condition: An ideal A satisfies the $\sum_p$-condition provided the following holds for arbitrary Banach spaces E, F and G_n $(n = 1, 2,...)$: If $T_1 \in L(E, (\sum G_n)_p)$, $T_2 \in L((\sum G_n)_p, F)$, and $T_2 P_n T_1 \in A(E, F)$, then $T_2 T_1 \in A(E, F)$.

(iii) Let $(x_k) \subset E$ be bounded. By assumption, $T_2 P_n T_1$ is a Banach–Saks operator. According to a result of Erdös and Magidor (cf. also [7, Corollary 2]) we can find for each n a subsequence (x_k') of (x_k) such that all subsequences of $(T_2 P_n T_1 x_k')$ are Cesaro convergent. Using this and a diagonal argument, we can extract a subsequence (x_k'') of (x_k') such that $(T_2 P_n T_1 x_k'')$ is Cesaro convergent for all n simultaneously. Since l_p $(1 < p < \infty)$ has the Banach–Saks property, we may also assume (x_k'') to be chosen in such a way that the sequence (ν_k), where

$$\nu_k = (\| P_1 T_1 x_k'' \|, \| P_2 T_1 x_k'' \|,...) \in l_p ,$$

is Cesaro convergent in l_p. From this we derive immediately that for each $\epsilon > 0$ there is a number N such that for $m > N$ the following holds:

$$\sum_{n=N+1}^{\infty} \left(\frac{1}{m} \sum_{k=1}^{m} \| P_n T_1 x_k'' \| \right)^p \leqslant \epsilon^p.$$

Consequently,

$$\left(\sum_{n=N+1}^{\infty} \left\| \frac{1}{m} \sum_{k=1}^{m} P_n T_1 x_k'' \right\|^p \right)^{1/p} \leqslant \epsilon,$$

and we get, applying the operator T_2,

$$\left\| \frac{1}{m} \sum_{k=1}^{m} \left(T_2 \sum_{n=N+1}^{\infty} P_n T_1 x_k'' \right) \right\| \leqslant \epsilon \| T_2 \|.$$

On the other hand, by the choice of (x_k''), the sequence

$$\frac{1}{m} \sum_{k=1}^{m} T_2 \sum_{n=1}^{N} P_n T_1 x_k''$$

converges in F for $m \to \infty$. This yields that

$$\frac{1}{m} \sum_{k=1}^{m} T_2 T_1 x_k''$$

converges in F, which proves (iii).

(iv) Let $S \in L(F, L_\infty(\Omega, \mu))$ and denote $S_2 = ST_2$. By assumption, there exist measurable functions $a_n(\omega)$ such that

$$S_2 P_n T_1 x = \langle x, a_n(\cdot) \rangle.$$

Let (N_n) be an arbitrary strictly increasing sequence of natural numbers, and denote

$$b_n(\omega) = \sum_{k=N_{2n-1}}^{N_{2n}} a_k(\omega).$$

Furthermore, let x_n $(n = 1,..., M)$ be arbitrary elements of E. Then

$$\left\| \sum_{n=1}^{M} \langle x_n, b_n(\omega) \rangle \right\|_{L_\infty(\Omega,\mu)}$$

$$= \left\| \sum_{n=1}^{M} S_2 \sum_{k=N_{2n-1}}^{N_{2n}} P_k T_1 x_n \right\|$$

$$\leqslant \| S_2 \| \left(\sum_{n=1}^{M} \| T_1 x_n \|^p \right)^{1/p} \leqslant \| S_2 \| \| T_1 \| \left(\sum_{n=1}^{M} \| x_n \|^p \right)^{1/p}$$

The b_n's are essentially separably valued. Therefore there exists a countable set $\Gamma \subset E$ which is norming for almost all values of b_n. Excluding a countable number of sets of μ-measure zero, we get for almost all $\omega \in \Omega$,

$$\left| \sum_{n=1}^{M} \langle x_n, b_n(\omega) \rangle \right| \leqslant \| S_2 \| \| T_1 \| \left(\sum_{n=1}^{M} \| x_n \|^p \right)^{1/p},$$

simultaneously for all $(x_1, ..., x_M) \in \Gamma^M$. This implies

$$\left(\sum_{n=1}^{M} \| b_n(\omega) \|^q \right)^{1/q} \leqslant \| S_2 \| \| T_1 \| \qquad (\mu\text{-a.e.}),$$

where $p^{-1} + q^{-1} = 1$. Consequently,

$$\sum_{n=1}^{M} \int \| b_n(\omega) \|^q \, d\mu \leqslant \| S_2 \|^q \| T_1 \|^q.$$

Since the b_n's were arbitrary disjoint blocks of the sequence (a_n), it follows that

$$\lim_{M,N\to\infty} \int \left\| \sum_{k=M}^{N} a_k(\omega) \right\|^q d\mu = 0.$$

This shows that the series $\sum a_n$ converges in the norm of $L_q(E')$, and therefore also in probability. Consequently, the function $a(\omega) = \sum a_n(\omega)$ is measurable. On the other hand, we have

$$T_1 x = \lim_{N\to\infty} \sum_{n=1}^{N} P_n T_1 x,$$

hence

$$(S_2 T_1 x)(\omega) = \lim_{N\to\infty} \left\langle x, \sum_{n=1}^{N} a_n(\omega)\right\rangle \qquad (\mu\text{-a.e.}).$$

This finally implies

$$(ST_2T_1x)(\omega) = (S_2T_1x)(\omega) = \langle x, a(\omega)\rangle \qquad (\mu\text{-a.e.}),$$

which shows that the operator T_2T_1 is decomposing, concluding the proof.

It is an open problem whether every Radon–Nikodým operator factors through a Banach space with the Radon–Nikodým property. Theorem 2.3(iv) and the relation between decomposing and Radon–Nikodým operators mentioned above yield a partial answer, which concerns dual operators. This result has also been obtained independently by Reĭnov [16] and Stegall [18].

Corollary 2.4. *Every dual Radon–Nikodým operator factors through a dual space with the Radon–Nikodým property.*

Combining Theorem 2.3(iii, iv) with Proposition 2.2, we get the following result describing properties of general interpolation spaces. The first part answers a question of Beauzamy [4, p. 65].

Corollary 2.5. *Let (E_1, E_2) be an interpolation pair, let $I\colon E_1 \cap E_2 \to E_1 + E_2$ be the canonical embedding, and let $1 < p < \infty$, $0 < \theta < 1$.*

(i) *If I is a Banach–Saks operator, then $(E_1, E_2)_{\theta,p}$ has the Banach–Saks property.*

(ii) *If the dual of I is a Radon–Nikodým operator, then $(E_1, E_2)'_{\theta,p}$ has the Radon–Nikodým property.*

In conclusion, we give two examples showing that Theorem 2.1 cannot be improved, in general. As already mentioned in Section I, the $\sum_p$-condition cannot be omitted. We show here that the assumption on both injectivity and surjectivity is essential.

Proposition 2.6. *There is an injective operator ideal A_0 and a surjective* *operator ideal B_0 which both satisfy the $\sum_p$-condition for all p with $1 < p < \infty$ and are not idempotent, hence, do not possess the factorization property.*

Proof. Let us consider the following ideal Z of Grothendieck operators. $T \in Z(E, F)$ iff every weak-star convergent sequence $(g_n) \subset F'$ is mapped by T' into a weakly convergent sequence $(T'g_n) \subset E'$. We put $B_0 = Z \cap X$, where X is the ideal of separable operators. X and Z are surjective ideals. It can be verified (but we omit this standard procedure) that Z and X satisfy the $\sum_p$-condition for $1 < p < \infty$, and therefore B_0 satisfies it, as well.

Now

$$B_0 \circ B_0 = (Z \cap X) \circ (Z \cap X) \subset X \circ Z.$$

It is easily seen that $X \circ Z \subset W$, where W is the ideal of weakly compact operators. On the other hand, there are non-weakly-compact operators belonging to $Z \cap X$. Take, e.g., the isometric embedding of l_1 into l_∞ (l_∞ possesses the Grothendieck property). This proves the surjective case.

The injective case can be seen by dualizing the above argument. Given an operator ideal A, we denote the class of all operators T with $T' \in A$ by A^{dual}. Now take $A_0 = Z^{\text{dual}} \cap X^{\text{dual}}$. This is an injective operator ideal which also satisfies the $\sum_p$-condition for $1 < p < \infty$. Again,

$$A_0 \circ A_0 \subset Z^{\text{dual}} \circ X^{\text{dual}} \subset W^{\text{dual}} = W.$$

However, the canonical surjection of l_1 onto c_0 belongs to A_0, but not to W.

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